Some Formulae for the Number of SDRs and Symbolic Representations

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If $(A_i: 1 \leq i \leq n)$ is a family of n finite sets, then two expressions for the number of SDRs for this family are obtained in terms of the cardinalities of unions and symmetric differences respectively. The proof makes use of the already known expression in terms of intersections. For this known expression is given a symbolic representation in the form of a determinant defined in a special manner. Also a representation, again as a determinant, for the number of SDRs in terms of the complements of the sets A_i is given.

1. INTRODUCTION

If X is a finite set, then $|X|$ denotes its cardinality and $\mathbb{P}(X)$ denotes the set of all partitions of X. For each positive integer n, let J_n denote the set $\{1, 2, \ldots, n\}$ of integers from 1 to n. If $(A_i: i \in J_n)$ is a family of n finite sets, then $\mathbb{D}(A_1 \times \cdots \times A_n)$ or $\mathbb{D}(\times_{i \in J_n} A_i)$ denotes the set of elements of the cartesian product $A_1 \times \cdots \times A_n$ with distinct coordinates. In other words $\mathbb{D}(A_1 \times \cdots \times A_n)$ consists of all SDRs for the family $(A_i: i \in J_n)$.

The number of SDRs in terms of the intersections of the sets A_i is given by

$$|\mathbb{D}(A_1 \times \cdots \times A_n)| = \sum_{Q \in \mathbb{P}(J_n)} \prod_{C \in Q} (-1)^{|C|-1}(|C|-1)! \left| \bigcap_{i \in C} A_i \right|. \tag{1.1}$$

A proof of this by Möbius inversion is given in Graver and Watkins [1, Chapter XI, Section D, Proposition D 24].

We give two more formulae in unions and symmetric differences:

$$|\mathbb{D}(A_1 \times \cdots \times A_n)| = \sum_{Q \in \mathbb{P}(J_n)} \prod_{C \in Q} (|C|-1)! \left(\left| \bigcup_{i \in C} A_i \right| - (n-1) \right), \tag{1.2}$$

$$|\mathbb{D}(A_1 \times \cdots \times A_n)| = \frac{1}{2^n} \sum_{Q \in \mathbb{P}(J_n)} \prod_{C \in Q} (|C|-1)! \left(2 \left| \underset{i \in C}{\Delta} A_i \right| - (n-1) \right). \tag{1.3}$$

Our object is to prove (1.2) and (1.3) which have a close resemblance to (1.1). We devote Sections 2 and 3 to this.

We shall make use of (1.1) itself (in more than one way) and certain combinatorial identities to derive (1.2) and (1.3).

In Section 4, we give two symbolic representations in the form of determinants for the number of SDRs in terms of intersections of the given sets and their complements respectively.

2. PRELIMINARIES

LEMMA 2.1. *If m is a real (or complex) number, then*

$$\sum_{Q \in \mathbb{P}(J_n)} \left(\prod_{C \in Q} (|C|-1)! \right) (-1)^{|Q|} m^{|Q|} = (-1)^n m(m-1) \cdots (m-n+1). \tag{2.1}$$

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PROOF. In formula (1.1), let $A_1 = A_2 = \cdots = A_n = S$, where S is an m-set. We have

$$|\mathbb{D}(\underbrace{S \times S \times \cdots \times S}_{n \text{ times}})| = \sum_{Q \in \mathbb{P}(J_n)} \Big(\prod_{C \in Q} (-1)^{|C|-1}(|C|-1)! \Big) m^{|Q|}$$

$$= \sum_{Q \in \mathbb{P}(J_n)} (-1)^{n-|Q|} \Big(\prod_{C \in Q} (|C|-1)! \Big) m^{|Q|}.$$

But the l.h.s. is plainly $(m!/(m-n)!)$ for $m \geqslant n$. Multiplying both sides by $(-1)^n$, (2.1) follows for all integers $m \geqslant n$. Since both sides of (2.1) are polynomials in m of degree n, it follows that (2.1) is true for all values of m, real or complex.

LEMMA 2.2. *Let $l_1, l_2, \ldots, l_k$ be k fixed positive integers whose sum is s and $f: J_n \to J_k$ be an arbitrary function. Let $|f^{-1}(i)|$ denote the number of elements in J_n whose image under f is i, for each $i \in J_k$. (Thus $|f^{-1}(i)| \geqslant 0$). Then*

$$\sum_{f \in J_k^{J_n}} (l_1 + |f^{-1}(1)|)!(l_2 + |f^{-1}(2)|)! \cdots (l_k + |f^{-1}(k)|)!$$
$$= l_1! l_2! \cdots l_k! \frac{(n+s+k-1)!}{(s+k-1)!}. \tag{2.2}$$

PROOF. The number of ways in which $|f^{-1}(1)|, |f^{-1}(2)|, \ldots, |f^{-1}(k)|$ take the values $P_1, P_2, \ldots, P_k$ respectively where $P_1 + P_2 + \cdots + P_k = n$, $P_i \geqslant 0$ is plainly $(n!/P_1!P_2! \cdots P_k!)$. Hence, the l.h.s. of (2.2) equals

$$\sum (l_1 + P_1)! \cdots (l_k + P_k)! \frac{n!}{P_1! \cdots P_k!},$$

where $\sum$ runs over all ordered k-tuples $(P_1, \ldots, P_k)$ such that $P_i \geqslant 0$ and $P_1 + \cdots + P_k = n$.

This expression is equal to

$$l_1! l_2! \cdots l_k! n! \sum \binom{l_1 + P_1}{P_1} \cdots \binom{l_k + P_k}{P_k} = l_1! l_2! \cdots l_k! n! \binom{l_1 + l_2 + \cdots + l_k + n + k - 1}{n}.$$

The last equality follows by equating coefficients of x^n on both sides of the identity

$$(1-x)^{-(l_1 + \cdots + l_k + k)} = (1-x)^{-(l_1+1)} \cdots (1-x)^{-(l_k+1)}.$$

Hence, l.h.s. of $(2.2) = l_1! l_2! \cdots l_k! n! \binom{n+s+k-1}{n} = l_1! l_2! \cdots l_k! (n+s+k-1)!/(s+k-1)! =$ r.h.s. of (2.2). This completes the proof.

We shall need the following two simple relations expressing the cardinality of an intersection of sets in terms of the cardinalities of unions and symmetric differences.

$$\Big| \bigcap_{i \in J_n} A_i \Big| = \sum_{\substack{T \subset J_n \\ T \neq \varnothing}} (-1)^{|T|-1} \Big| \bigcup_{i \in T} A_i \Big|, \tag{2.3}$$

$$\Big| \bigcap_{i \in J_n} A_i \Big| = \frac{1}{2^{n-1}} \sum_{\substack{T \subset J_n \\ T \neq \varnothing}} (-1)^{|T|-1} \Big| \mathop{\Delta}_{i \in T} A_i \Big|, \tag{2.4}$$

where $(A_i : i \in J_n)$ is a family of n finite sets.

3. Proofs of (1.2) and (1.3)

Proof of (1.2). We first consider the r.h.s. of (1.2). Let $S \subset J_n$, $S \neq \emptyset$ and $P \in \mathbb{P}(S)$. What is the coefficient of $\prod_{C\in P}|\bigcup_{i\in C} A_i|$ in the r.h.s. of (1.2)? Since $\prod_{C\in P}(|C|-1)!$ occurs in every term as a factor of the coefficient of this, we have $\prod_{C\in P}(|C|-1)!$ as an obvious factor of the required coefficient. Apart from this we have other terms occurring in the coefficients which have to be summed up. For this, consider a typical partition $Q \in \mathbb{P}(J_n \backslash S)$. Then the remaining contribution would be

$$\sum_{Q\in\mathbb{P}(J_n\backslash S)} \left(\prod_{C\in Q}(|C|-1)!\right)(-1)^{|Q|}(n-1)^{|Q|}.$$

By Lemma 2.1, this is

$$(-1)^{|J_n\backslash S|}\frac{(n-1)!}{((n-1)-|J_n\backslash S|)!} = \frac{(-1^{n-|S|}(n-1)!}{(|S|-1)!}.$$

Hence the coefficient of $\prod_{C\in P}|\cup_{i\in C} A_i|$ is

$$(-1)^{n-|S|}\frac{(n-1)!}{(|S|-1)!}\prod_{C\in P}(|C|-1)!. \tag{3.1}$$

Next we consider the l.h.s. of (1.2). By (1.1) and (2.3), the l.h.s. of (1.2) =

$$\textstyle\sum_{Q\in\mathbb{P}(J_n)}\prod_{C\in Q}(-1)^{|C|-1}(|C|-1)!\,(\sum_{T\subset C, T\neq\emptyset}|\bigcup_{i\in T}A_i|(-1)^{|T|-1}).$$

Again let $S \subset J_n$, $S \neq \emptyset$ and $P \in \mathbb{P}(S)$. We wish to find the coefficient of $\prod_{C\in P}|\bigcup_{i\in C} A_i|$ in the last expression. A little consideration shows that we may proceed as follows.

We distribute the elements of $J_n \backslash S$ to the blocks of P at random (some blocks may receive none) to get a partition of J_n itself into $|P|$ blocks. Each such augmented partition gives rise to the term $\prod_{C\in P}|\bigcup_{i\in C} A_i|$ along with certain integral coefficients. All this can be rigorously put thus: take any function $f: J_n\backslash S \to P$. For each $C \in P$ define $C_f = C \cup f^{-1}(C)$. Then $\{C_f : C \in P\}$ is a partition of J_n and the coefficient of $\prod_{C\in P}|\cup_{i\in C} A_i|$ is equal to

$$\begin{aligned}\left(\sum_{f\in P^{J_n\backslash S}}\prod_{C\in P}(-1)^{|C_f|-1}(|C_f|-1)!\right)&\prod_{C\in P}(-1)^{|C|-1}\\ &=(-1)^{n-|P|}(-1)^{|S|-|P|}\sum_{f\in P^{J_n\backslash S}}\prod_{C\in P}(|C_f|-1)!\\ &=(-1)^{n-|S|}\sum_{f\in P^{J_n\backslash S}}\prod_{C\in P}(|C_f|-1)!,\end{aligned}$$

since $\sum_{C\in P}|C_f| = n$ and $\sum_{C\in P}|C| = |S|$. But by Lemma (2.2), this last expression is equal to

$$(-1)^{n-|S|}\left[\prod_{C\in P}(|C|-1)!\right]\frac{(n-1)!}{(|S|-1)!}. \tag{3.2}$$

From (3.1) and (3.2), we see that the coefficients of $\prod_{C\in P}|\bigcup_{i\in C} A_i|$ have agreed for each $S \subset J_n$, $S \neq \emptyset$ and for each $P \in \mathbb{P}(S)$. Further the constant in the r.h.s. of (1.2) can easily be seen to be zero by another application of Lemma 2.1. Hence we have proved (1.2).

Proof of (1.3). Since this proof also runs on lines similar to the above, we omit the details and only outline the method.

Application of Lemma 2.1 gives the coefficient of $\prod_{C\in P}|\Delta_{i\in C}A_i|$, for any $S\subset J_n$, $S\neq\varnothing$ and any $P\in\mathbb{P}(S)$, in the r.h.s. of (1.3) to be

$$(-1)^{n-|S|}\frac{(n-1)!}{(|S|-1)!}\cdot\frac{1}{2^n}\cdot\prod_{C\in P}[2(|C|-1)!] \tag{3.3}$$

and the constant term in the same is zero as before.

By (1.1) and (2.4), the l.h.s. of (1.3) can be written as

$$\sum_{Q\in\mathbb{P}(J_n)}\prod_{C\in Q}(-1)^{|C|-1}(|C|-1)!\left(\frac{1}{2^{|C|-1}}\sum_{\substack{T\subset C\\ T\neq\varnothing}}\left|\Delta_{i\in T}A_i\right|(-1)^{|T|-1}\right).$$

Application of Lemma 2.2 now gives the coefficient of $\prod_{C\in P}|\Delta_{i\in C}A_i|$ to be the same as (3.3).

Hence (1.3) is proved.

Incidentally, we have proved the following corollary.

COROLLARY 3.1.

(a) $$|\mathbb{D}(A_1\times\cdots\times A_n)|=(n-1)!\sum_{\substack{S\subset J_n\\ S\neq\varnothing}}\frac{(-1)^{n-|S|}}{(|S|-1)!}\sum_{P\in\mathbb{P}(S)}\prod_{C\in P}(|C|-1)!\left|\bigcup_{i\in C}A_i\right| \tag{3.4}$$

and

(b) $$|\mathbb{D}(A_1\times\cdots\times A_n)|=\frac{(n-1)!}{2^n}\sum_{\substack{S\subset J_n\\ S\neq\varnothing}}\frac{(-1)^{n-|S|}}{(|S|-1)!}\sum_{P\in\mathbb{P}(S)}\prod_{C\in P}[2(|C|-1)!]\left|\Delta_{i\in C}A_i\right|. \tag{3.5}$$

It will be interesting to obtain (1.2) and (1.3) or (3.4) and (3.5) by the principle of Möbius inversion.

Note that (1.3) may be written in the form:

$$|\mathbb{D}(A_1\times\cdots\times A_n)|=2^{-n}\sum_{Q\in\mathbb{P}(J_n)}\prod_{C\in Q}(2(|C|-1)!)\left(\left|\Delta_{i\in C}A_i\right|-\binom{n-1}{2}\right). \tag{3.6}$$

We can easily establish the following statements involving the coefficients in the right-hand sides of (1.1), (1.2), (3.4), (3.5) and (3.6) when written fully (with no $\sum$ occurring in the expansions).

LEMMA 3.1

(a) *The algebraic sum of the coefficients in the r.h.s. of* (1.1) *is* 1 *for* $n=1$ *and zero for* $n\geq 2$, *whereas the sum of their absolute values is* $n!$ *for all* n.

(b) *The sum of the coefficients in the r.h.s. of* (1.2) *is* $n!$ *and it is* $(n+1)!/2^n$ *in the r.h.s. of* (3.6) *for each* n.

(c) *The algebraic sum of the coefficients in the r.h.s. of* (3.4) *is* 1 *for* $n=1$ *and zero for* $n\geq 2$, *whereas the sum of their absolute values is* $2^{n-1}n!$ *for each* n.

(d) *The algebraic sum of the coefficients in the r.h.s. of* (3.5) *is* 1 *and* $\frac{1}{2}$ *for* $n=1$ *and* 2 *respectively and zero for* $n\geq 3$, *whereas the sum of their absolute values is* $n!((n+3)/4)$ *for all* n.

PROOF

(a) Put $m=1$ and $m=-1$ respectively in Lemma (2.1).

(b) Put $m=-1$ and $m=-2$ respectively in Lemma (2.1).

(c) Use first part of (b) and simple binomial identities.

(d) Use second part of (b) and binomial identities.

We conclude this section by proving a simple corollary of Lemma 2.1, though this is a little out of the context.

COROLLARY 3.2. *Let $s(n, k)$ denote the sum of products of $1, 2, \ldots, n$ taken k at a time and $P(n, k)$ denote the number of permutations of $1, 2, \ldots, n$ into k disjoint cycles of length ≥ 1. Then*

$$s(n, k) = P(n+1, n-k+1).$$

PROOF. Consider Equation (2.1) which is true for any m. The coefficient of m^{n-k} on the r.h.s. is $(-1)^{n+k}s(n-1, k)$. The coefficient of m^{n-k} on the l.h.s. is

$$\sum_{Q \in \mathbb{P}(J_n)} \left(\prod_{C \in Q} (|C|-1)! \right) (-1)^{n-k}.$$

But we know that to find the number of permutations of $1, 2, \ldots n$ into k disjoint cycles of length ≥ 1, we could first partition $1, 2, \ldots, n$ into k blocks and then form cycles from each block. If C is one such block, the number of cycles that can be formed from it is exactly $(|C|-1)!$. Hence

$$P(n, n-k) = \sum_{\substack{Q \in \mathbb{P}(J_n) \\ |Q| = n-k}} \left(\prod_{C \in Q} (|C|-1)! \right).$$

Therefore, $(-1)^{n+k}s(n-1, k) = (-1)^{n-k}P(n, n-k)$. Increasing n by 1, we get the desired result.

4. SYMBOLIC REPRESENTATIONS

A. Statement (a) of Lemma 3.1 suggests that the r.h.s. of (1.1) may have a "determinant" representation.

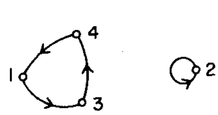
Though this is not the case in the strict sense, we can still express the r.h.s. of (1.1) in a symbolic form which is very much like a determinant. For this we define a few symbols. In what follows, intersection of sets is denoted by juxtaposition.

Let $(A_i : i \in J_n)$ be a family of n finite sets. Consider a "circle" product of the type

$$(A_{a_1}, A_{b_1}) \circ (A_{a_2}, A_{b_2}) \circ \cdots \circ (A_{a_n}, A_{b_n}) \tag{4.1}$$

of n ordered pairs whose coordinates are the indexed sets of the family and where $a_1, a_2, \ldots, a_n$ are the numbers $1, 2, \ldots, n$ in some order and so are $b_1, b_2, \ldots, b_n$. Construct a directed graph G with vertex set J_n and n directed edges (or loops) (a_i, b_i), $1 \leq i \leq n$. The components of this digraph G give rise to a partition P of J_n. Define now the circle product (4.1) to be the ordinary product $\prod_{C \in P} |\bigcap_{i \in C} A_i|$, with the help of the partition P so obtained. It is clear that the $\circ$ operation is commutative and hence in (4.1), we may take $a_i = i$ without loss of generality. As an example for $n = 4$, we have

$(A_1, A_3) \circ (A_2, A_2) \circ (A_3, A_4) \circ (A_4, A_1) = |A_1 A_3 A_4| \cdot |A_2|$ and the corresponding digraph is as given:



It is easy to check that

$$\left| \begin{matrix} (A_1, A_1) & (A_1, A_2) \\ (A_2, A_1) & (A_2, A_2) \end{matrix} \right| = |A_1| \cdot |A_2| - |A_1 A_2| = |\mathbb{D}(A_1 \times A_2)|$$

and

$$\left|\begin{matrix} (A_1, A_1) & (A_1, A_2) & (A_1, A_3) \\ (A_2, A_1) & (A_2, A_2) & (A_2, A_3) \\ (A_3, A_1) & (A_3, A_2) & (A_3, A_3) \end{matrix}\right|^{\circ}$$

$$= |A_1| \cdot |A_2| \cdot |A_3| - |A_1| \cdot |A_2A_3| - |A_2| \cdot |A_1A_3| - |A_3| \cdot |A_1A_2| + 2|A_1A_2A_3|$$
$$= |\mathbb{D}(A_1 \times A_2 \times A_3)|,$$

where we expand each determinant in the usual manner, but take the circle product in each of the terms and then convert each circle product into an ordinary product by the above rule. We shall call such determinants circle determinants.

In general we have the following proposition.

PROPOSITION 4.1. *If $(A_i : i \in J_n)$ is a family of n finite sets, then*

$$|\mathbb{D}(A_1 \times \cdots \times A_n)| = \left|\begin{matrix} (A_1, A_1) & (A_1, A_2)\ldots & (A_1, A_n) \\ (A_2, A_1) & (A_2, A_2)\ldots & (A_2, A_n) \\ \cdots & \cdots & \cdots \\ (A_n, A_1) & (A_n, A_2)\ldots & (A_n, A_n) \end{matrix}\right|^{\circ} \tag{4.2}$$

where the r.h.s. determinant is expanded as described above.

PROOF. Let $\sigma = (a_1a_2 \cdots a_k)(b_1b_2 \cdots b_l) \cdots (h_1h_2 \cdots h_s)$, $k, l, \ldots, s \geqslant 1$, be a typical element of S_n, the symmetric group of order n, consisting of the permutations of $1, 2, \ldots, n$. The r.h.s. of (4.2) can be expanded in the form

$$\sum_{\sigma \in S_n} \varepsilon_\sigma (A_{a_1}, A_{a_2}) \circ (A_{a_2}, A_{a_3}) \circ \cdots \circ (A_{a_k}, A_{a_1}) \circ (A_{b_1}, A_{b_2}) \circ (A_{b_2}, A_{b_3}) \circ$$
$$\cdots \circ (A_{b_l}, A_{b_1}) \circ \cdots \circ (A_{h_1}, A_{h_2}) \circ (A_{h_3}, A_{h_3}) \circ \cdots \circ (A_{h_s}, A_{h_1})$$
$$= \sum_{\sigma \in S_n} \varepsilon_\sigma |A_{a_1}A_{a_2} \cdots A_{a_k}| \cdot |A_{b_1}A_{b_2} \cdots A_{b_l}| \cdot \cdots \cdot |A_{h_1}A_{h_2} \cdots A_{h_s}|$$

by the above rule. Here ε_σ is 1 or -1 according to whether σ is even or odd. Now we consider the number of $\sigma \in S_n$ which give rise to the same product. The number of permutations $\sigma \in S_n$ with pre-assigned lengths of cycles, namely, $k, l, \ldots, s \geqslant 1$ and pre-assigned numbers occurring in these cycles is clearly

$$(k-1)!(l-1)! \cdots (s-1)!.$$

Further $\varepsilon_\sigma = (-1)^{(k-1)+l-1 \cdots +(s-1)}$, since these cycles are products of $(k-1)$, $(l-1), \ldots, (s-1)$ transpositions respectively. Also these are the only permutations in S_n that give rise to the same product. In other words, corresponding to the partition

$$\{\{a_1, \ldots, a_k\}, \{b_1, \ldots, b_l\}, \ldots, \{h_1, \ldots, h_s\}\}$$

of J_n we obtain the term (in the expansion of the determinant)

$$(-1)^{(k-1)+\cdots+(s-1)}(k-1)! \cdots (s-1)!|A_{a_1} \cdots A_{a_k}| \cdot \cdots \cdot |A_{h_1} \cdots A_{h_s}|$$

which is exactly a term in the r.h.s. of (1.1) for the same partition. This proves (4.2). We shall denote the matrix that corresponds to the above circle determinant by Δ, i.e. $\Delta = [(A_i, A_j)]$.

B. Consider n subsets $A_1, \ldots, A_n$ (not necessarily distinct) of a ground set X, with $|X| = m$. Let $A_1^c, \ldots, A_n^c$ be their complements in X. We wish to find $|\mathbb{D}(A_1^c \times \cdots \times A_n^c)|$

in terms of the cardinalities of $A_1, \ldots, A_n$ and their intersections. By the principle of inclusion and exclusion $|\mathbb{D}(A_1^c \times \cdots \times A_n^c)|$ can be written as

$$\frac{m!}{(m-n)!} - a_1 \frac{(m-1)!}{(m-n)!} + a_2 \frac{(m-2)!}{(m-n)!} - \cdots + (-1)^n a_n \frac{(m-n)!}{(m-n)!},$$

where a_r denotes the number of r "hits", i.e. choose r places $i_1, \ldots, i_r$ and fill these places with elements of $A_{i_1}, \ldots, A_{i_r}$ respectively such that they are all distinct. Then the remaining places (i.e. coordinates) are filled with elements of X such that all the n coordinates are distinct. In other words,

$$a_r = \sum_{\substack{S \subset J_n \\ |S| = r}} \left|\mathbb{D}\left(\bigtimes_{i \in S} A_i\right)\right|, \qquad 0 \leq r \leq n,$$

a_0 being defined as 1. Hence

$$\begin{aligned} |\mathbb{D}(A_1^c \times \cdots \times A_n^c)| &= \sum_{S \subset J_n} (-1)^{|S|} \left|\mathbb{D}\left(\bigtimes_{i \in S} A_i\right)\right| \frac{(m-|S|)!}{(m-n)!} \\ &= \frac{1}{(m-n)!} \sum_{S \subset J_n} (-1)^{|S|} \left|\mathbb{D}\left(\bigtimes_{i \in S} A_i\right)\right| (m-|S|)!. \end{aligned} \tag{4.3}$$

This also can be put in the determinant form thus:

$$\frac{\lambda^{m-n}}{(m-n)!} \begin{vmatrix} \lambda-(A_1, A_1) & -(A_1, A_2) & \cdots & -(A_1, A_n) \\ -(A_2, A_1) & \lambda-(A_2, A_2) & \cdots & -(A_2, A_n) \\ \cdots & \cdots & \cdots & \cdots \\ -(A_n, A_1) & -(A_n, A_2) & \cdots & \lambda-(A_n, A_n) \end{vmatrix} \tag{4.4}$$

where the determinant is to be first expanded as a polynomial in λ with coefficients themselves as sums of "circle determinants" of the type described in A in this section (coefficient of λ^{n-k} is the sum of $\binom{n}{k}$ principal minor determinants of order k, $1 \leq k \leq n$, which are all circle determinants); then each determinant is expanded as in A and λ^k is replaced by $k!$ (after multiplying by the outside factor $\lambda^{m-n}!$). [If $|X|$ is also n, the above determinant (4.4) can be compactly put in the form (since $m = n$), $|\lambda I - \Delta|$, where Δ is as in A.]. All these operations lead precisely to the r.h.s. of (4.3). Thus the r.h.s. of (4.3) can be represented symbolically by (4.4).

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